

1. (12.4) Prove that the sequence  $\{\frac{1}{1+n^2}\}_{n \in \mathbb{N}}$  converges to 0.
2. (12.7) By definition,  $\lim_{n \rightarrow \infty} a_n = L$  if for every  $\epsilon > 0$ , there exists a positive integer  $N$  such that if  $n$  is an integer with  $n > N$ , then  $|a_n - L| < \epsilon$ . By taking the negation of this definition, write out the meaning of  $\lim_{n \rightarrow \infty} a_n \neq L$  using quantifiers. Then write out the meaning of  $\{a_n\}$  diverges using quantifiers.
3. (12.8) Show that the sequence  $\{n^4\}_{n \in \mathbb{N}}$  diverges to infinity.
4. (12.10)
  - (a) Prove that  $1 + \frac{1}{2} + \cdots + \frac{1}{n} < 2\sqrt{n}$  for every positive integer  $n$ . (you may try induction)
  - (b) Let  $s_n = \frac{1}{n} + \frac{1}{2n} + \cdots + \frac{1}{n^2}$  for each  $n \in \mathbb{N}$ . Prove that the sequence  $\{s_n\}$  converges to 0.
5. Mark True or False. Justify each answer. (you may use the limit laws if needed)
  - (a) If  $\{a_n\}$  converges to  $a$  and  $a_n > 0$  for all  $n \in \mathbb{N}$  then  $a > 0$ .
  - (b) If  $\{a_n\}$  and  $\{b_n\}$  are both divergent sequences, then  $\{a_n + b_n\}$  diverges.
  - (c) If  $\{a_n\}$  and  $\{b_n\}$  are both divergent sequences, then  $\{a_n b_n\}$  diverges.
  - (d) If  $\{a_n\}$  and  $\{a_n + b_n\}$  are both convergent sequences, then  $\{b_n\}$  converges.

6. Find the following limits

- (a)  $\lim_{n \rightarrow \infty} \frac{3n^2 + 4n}{7n^2 - 5n}$
- (b)  $\lim_{n \rightarrow \infty} \frac{\sin n}{2n + 1}$
- (c)  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 1} - n)$ .

7. (12.16)

- (a) Prove that if  $\sum_{k=1}^{\infty} a_k$  is a convergent series, then  $\lim_{n \rightarrow \infty} a_n = 0$ .
- (b) Show that the converse of the result in (a) is false.

8. Determine whether each series converges or diverges. Justify your answer.

- (a)  $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$
  - (b)  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
  - (c)  $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$
  - (d)  $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$
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9. Mark each statement True or False. Justify your answer.

- (a)  $\sum_{n=1}^{\infty} a_n$  converges iff  $\lim_{n \rightarrow \infty} a_n = 0$ .
- (b) The geometric series  $\sum_{n=0}^{\infty} r^n$  converges iff  $r < 1$ .
- (c)  $\sum_{n=1}^{\infty} a_n$  converges if the sequence  $\{a_n\}$  is bounded.
- (d)  $\sum_{n=1}^{\infty} a_n$  converges if the sequence  $\{s_n\}$  where  $s_n = a_1 + \dots + a_n$  is bounded.

**The following questions are optional and will not be marked.**

10. For each subset of  $\mathbf{R}$ , find its maximum and supremum, minimum and infimum, if they exist. Otherwise write "none".

- (a)  $[0, 2)$
- (b)  $\{r \in \mathbf{Q} : 3 < r^2 \leq 4\}$
- (c)  $\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$

11. Let  $S$  be a nonempty subset of  $\mathbf{R}$  and let  $m = \sup S$ . Prove that  $m \in S$  iff  $m = \max S$ .

12. Let  $S$  be a nonempty bounded subset of  $\mathbf{R}$ . Show  $\inf S \leq \sup S$ . What can you conclude if  $\inf S = \sup S$ ?