Problems are mostly (but not all) from Chapter 2 of the 3rd edition of the text. Problems coming from the textbook have their problem number listed below.

If you are not using the 3rd edition, be careful — question numbers may not agree.

1. **(2.4)** Consider the open sentence $P(x) : x(x-1) = 6$ over the domain $\mathbb{R}$.
   (a) For what values of x is $P(x)$ a true statement?

   Solution: $x \in \{-2, 3\}$

   (b) For what values of x is $P(x)$ a false statement?

   Solution: $x \in \mathbb{R} - \{-2, 3\}$

2. **(2.14)** State the negation of each of the following statements.
   (a) At least two of my library books are overdue.

   Solution: Fewer than two of my library books are overdue.

   (c) No one expected that to happen.

   Solution: Someone expected that to happen.

   (e) It's surprising that there were two students who received the same exam score.

   Solution: It's not surprising that there were two students who received the same exam score.

3. **(2.18)** Let $S = \{1, 2, 3, 4, 5, 6\}$ and let

$$P(A) : A \cap \{2, 4, 6\} = \emptyset \quad \text{and} \quad Q(A) : A \neq \emptyset$$

   be open sentences over the domain $\mathcal{P}(S)$.

   (a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.

   Solution: $A \in \mathcal{P}(\{1, 3, 5\}) - \emptyset$. Can also be written as

$$A \in \{\{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$$

   (b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \vee (\sim Q(A))$ is true.

   Solution: $A \in \mathcal{P}(\{1, 3, 5\})$

4. **(2.23)** Suppose that $\{S_1, S_2\}$ is a partition of the set S and $x \in S$. Which of the following are true? (Multiple are possible)
   (a) If we know that $x \notin S_1$, then x must belong to S_2. **True.**
   (b) It's possible that $x \notin S_1$ and $x \notin S_2$. **False.**
   (c) $x \notin S_1$ or $x \notin S_2$. **True.**
   (d) $x \in S_1$ or $x \in S_2$. **True.**

(e) It's possible that $x \in S_1$ and $x \in S_2$. **False.**

5. **(2.34)** Each of the following describes an implication. Write the implication in the form 'if, then.'

   (a) Any point on the straight line with equation $2y + x - 3 = 0$ whose x-coordinate is an integer also has an integer for its y-coordinate.

   Solution: If $(x, y) \in \mathbb{R}^2$ is such that $2y + x - 3 = 0$ and x is an integer, then y is an integer.

   OR If x is an integer and y is a real number such that $2y + x - 3 = 0$, then y is an integer.

   (b) The square of any odd integer is odd.

   Solution: If n is an odd integer, then n^2 is odd.

   (c) Let $n \in \mathbb{Z}$. Whenever $3n + 7$ is even, n is odd.

   Solution: If n is an integer such that $3n + 7$ is even, then n is odd.

6. **(2.42)** Determine all values of n in the domain $S = \{2, 3, 4\}$ for which the following is a true statement: The integer $\frac{n(n-1)}{2}$ is odd if and only if the integer $\frac{n(n+1)}{2}$ is even.

   Solution: In order for the statement to holds true, either $\frac{n(n-1)}{2}$ is odd and $\frac{n(n+1)}{2}$ is even, or $\frac{n(n-1)}{2}$ is even and $\frac{n(n+1)}{2}$ is odd. $\frac{1 \cdot 2}{2} = 1$, $\frac{2 \cdot 3}{2} = 3$, $\frac{3 \cdot 4}{2} = 6$, $\frac{4 \cdot 5}{2} = 10$. So the statement is true for $n = 3$, and false for $n = 2, 4$.

7. **(2.54)** For statements P and Q, show that $((\sim Q) \implies (P \wedge (\sim P)))$ and Q are logically equivalent.

   Solution:

$$\begin{aligned} ((\sim Q) \implies (P \wedge (\sim P))) &\equiv (\sim (\sim Q) \vee (P \wedge (\sim P))) \\ &\equiv Q \vee (P \wedge (\sim P)) \\ &\equiv Q \end{aligned}$$

   the last step is because $P \wedge (\sim P)$ is a contradiction.

8. **(2.60)** Consider the implication: If x and y are even, then xy is even.

   (c) State the implication as a disjunction. (In particular, without the use of the word 'and' or the symbol $\wedge$.)

   Solution: Let's convert the statement into propositional logic.

$$((x \text{ is even}) \wedge (y \text{ is even})) \implies (xy \text{ is even})$$

$$\equiv \sim ((x \text{ is even}) \wedge (y \text{ is even})) \vee (xy \text{ is even})$$
$$\equiv (x \text{ is not even}) \vee (y \text{ is not even}) \vee (xy \text{ is even})$$

9. **(2.75)** Consider the quantified statement

For every $s \in S$ and every $t \in T$, $st - 2$ is prime.

where the domain of the variables s and t is $S = T = \{3, 5, 11\}$.

(a) Express this quantified statement in symbols.

Solution: $\forall s \in S, \forall t \in T, st - 2$ is prime.[1]

(b) Is the quantified statement in (a) true or false? Explain.

Solution: It is false, because the statement is not true for all pairs (s, t). If we take $s = t = 11$, we get $st - 2 = 119 = 7 \cdot 17$.

(c) Express the negation of the quantified statement in (a) in symbols.

Solution: $\exists s \in S, \exists t \in T, st - 2$ is not prime

10. **(2.86)** Given the implication $(Q \vee R) \implies (\sim P)$ is false and Q is false, determine the truth values of R and P.

Solution: R and P are both true, here is an argument for why. The negation of the implication, $(Q \vee R) \wedge \sim (\sim P)$, is true. Therefore, $\sim (\sim P) \equiv P$ is true, and $Q \vee R$ is true. Since Q is false, R is therefore true.

11. Consider the open sentence $P(a, b) : |a - b| < 2$ where the domain of a is $A = \{2, 5, 8\}$ and the domain of b is $B = \{3, 4, 7\}$.

(a) State the quantified statement $\forall a \in A, \exists b \in B, P(a, b)$ in words.

Solution: For each a in the set A, there exists some b in the set B such that $|a - b| < 2$.

(b) Prove that the quantified statement $\forall a \in A, \exists b \in B, P(a, b)$ is true.

Solution: We must check that for each a, there is a b such that $|a-b| < 2$. When $a = 2$, let $b = 3$. When $a = 5$, let $b = 4$. And when $a = 8$, let $b = 7$.

(c) Prove that the quantified statement $\exists b \in B, \forall a \in A, P(a, b)$ is false.

Solution: We can prove that the negation,

$$\sim (\exists b \in B, \forall a \in A, P(a, b))$$

$$\equiv \forall b \in B, \exists a \in A, (\sim P(a, b))$$

is true. We must show that for each b, there is an a such that $|a - b| \geq 2$. When $b = 3$, $a = 5$ (or $a = 8$). When $b = 4$, $a = 2$ (or $a = 8$). And when $b = 7$, $a = 2$ (or $a = 5$).

[1] If you want to know how to say that a natural number is prime, here is one way to say it:

$$\sim (\exists a \in \mathbb{N}, \exists b \in \mathbb{N}, (a > 1) \wedge (b > 1) \wedge (ab = n))$$