5.23 Prove there is no integer a such that $a \equiv 5 \pmod{14}$ and $a \equiv 3 \pmod{21}$.

5.55 Prove that there do not exist positive integers a and n such that $a^2 + 3 = 3^n$.

3. Carefully read the proof (from the class notes or from the book) that $\sqrt{2}$ is irrational. Then prove the following.

   (a) Prove that $\sqrt[3]{5}$ is irrational.

   (b) Prove that $\log_2(5)$ is irrational. (Remember that $\log_a(b)$ is the unique real number such that $a^{\log_a(b)} = b$.)

   (c) **(Bonus)** Prove that if n is a positive integer such that $n \geq 2$, then $\sqrt[n]{5/3}$ is irrational.

4. Prove that between any two real numbers, there are infinitely many rational numbers. (Hint: First prove as a lemma that between any two real numbers, there is at least one rational number.)

5. Consider the following proposition.

   Proposition 0.1. *Let x be any positive real number. Then for every positive real number y, there is a positive real number z such that $z^x > y$.*

   (a) The following is an invalid proof. Explain why it is invalid.

   Proof. Suppose that the statement is false. Then for some x, there is a positive real number y such that for every $z > 0$, $z^x \leq y$. Either $y > 1$ or $y \leq 1$: let us consider these two cases separately.

   - If $y > 1$, then set $z = y$ and $x = 2$. Then $z^x = y^2 > y$. But this contradicts the assumption that $z^x \leq y$, so we have a contradiction!
   - If $y \leq 1$, then set $z = 2$ and $x = 1$. Then $z^x = 2 > y$. But this contradicts the assumption that $z^x \leq y$, so we have a contradiction!

   In each case we reach a contradiction, and so our assumption was incorrect. Therefore, for every positive real number x and every positive real number y, there is a positive real number z such that $z^x > y$. □

   (b) Give a correct proof of the proposition.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that

$$\exists a, b \in \mathbb{R}, (a < b \quad \wedge \quad f(a) < f(b))$$

and

$$\exists c, d \in \mathbb{R}, (c < d \quad \wedge \quad f(c) > f(d))$$

Use the Intermediate Value Theorem to prove that

$$\exists x, y \in \mathbb{R}, (x < y \quad \wedge \quad f(x) = f(y))$$

Hint: Construct a continuous function $H(t)$ with the property that $H(0) = f(b) - f(a)$ and $H(1) = f(d) - f(c)$. Then use the Intermediate Value Theorem on $H(t)$ over the interval $[0, 1]$.