- There are 10 questions worth a total of 40.

1. 6 marks Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) reflexive but neither symmetric nor transitive
   (e) symmetric but neither reflexive nor transitive
   (f) transitive but neither reflexive nor symmetric

Solution:

(a) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)\}$

(b) $\{(1,1),(2,2),(3,3),(4,4),(1,2)\}$

(c) $\emptyset$

(d) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,3)\}$

(e) $\{(1,2),(2,1),(2,3),(3,2)\}$

(f) $\{(1,2),(2,3),(1,3)\}$

2. 4 marks Let $A = \{1\}$ and $B = \{1, 2\}$.
   (a) Determine all relations on A.
   (b) Determine all relations from A to B.
   (c) Determine all equivalence relations on B.

Solution:

(a) $\emptyset$ and $A \times A$

(b) $\emptyset$, $\{(1,1)\}$, $\{(1,2)\}$, $A \times B$

(c) To be reflexive, any equivalence relation R on B must contain $(1,1)$ and $(2,2)$. If R contains $(1,2)$, then by symmetry it also contains $(2,1)$. Similarly, if R contains $(2,1)$ then it also contains $(1,2)$. Therefore the only possibilities are $\{(1,1),(2,2)\}$ and $B \times B$. It is easy to check that these relations are equivalence relations.

3. 1 mark Let A be a non-empty set. Let $R = \emptyset$. Is R an equivalence relation on A: why or why not?

Solution:

No, because $\emptyset$ is not reflexive: since $A \neq \emptyset$ there is some $a \in A$ and $(a, a) \notin \emptyset$. (It is however trivially symmetric and transitive.)

4. 2 marks Let $n \geq 2$. Let $\mathbb{Z}_n = \{[0], [1], \ldots [n-1]\}$ denote the *integers modulo* n. Recall that the elements of $\mathbb{Z}_n$ are the distinct equivalence classes induced by the equivalence relation aRb if and only if $a \equiv b \bmod n$.

Show that multiplication in $\mathbb{Z}_n$ is well-defined.

Note: By Theorem 8.2, it suffices to verify that for all $a, b \in \mathbb{Z}_n$ and any $x \in [a]$ and $y \in [b]$ we have that $xy \in [ab]$.

Hint: see Result 4.11.

Solution:

Let $a, b \in \mathbb{Z}_n$. Let $x \in [a]$ and $y \in [b]$. Then $x \equiv a$ and $y \equiv b \bmod n$. By Result 4.11, $xy \equiv ab \bmod n$. Hence $xy \in [ab]$.

5. 6 marks Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. For $x, y \in \mathbb{R}$, write xRy if and only if $f(x) = y$.

(a) Verify that R is a relation on $\mathbb{R}$.

(b) Is R an equivalence relation: why or why not?

(c) Find the domain and range of R, $\text{dom}(R)$ and $\text{rng}(R)$.

(d) Describe the inverse relation R^{-1}.

(e) Sketch R and R^{-1}.

(f) Does R^{-1} correspond to the graph of a function: why or why not?

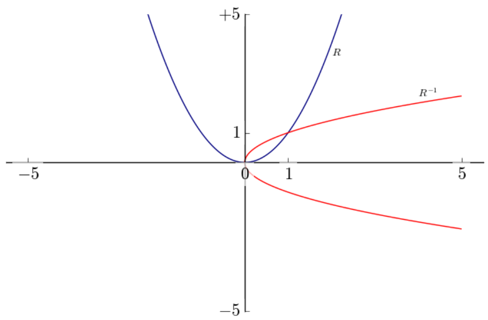
Solution:

(a) R is a relation on $\mathbb{R}$ since $R \subseteq \mathbb{R}^2$

(b) No, since $f(2) \neq 2$ and so $(2, 2) \notin R$.

(c) $\text{dom}(R) = \mathbb{R}$ and $\text{rng}(R) = \{y \in \mathbb{R} : x \geq 0\}$.

(d) $R^{-1} = \{(x, \pm\sqrt{x}) : x \geq 0\}$.



(e)

(f) No, since the plot of R^{-1} does not pass the vertical line test.

6. 6 marks Let $S = \{2^n : n \in \mathbb{Z}\}$. Define a relation R on $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$ by qRr if and only if $q/r \in S$.

   (a) Show that R is an equivalence relation.

   (b) Describe the elements of the equivalence classes $[1/2]$, $[2]$ and $[3]$.

Solution:

(a) R is reflexive, since for any $q \in \mathbb{Q}_+$, $q/q = 1 = 2^0 \in S$. R is symmetric, since if $q/r = 2^n$ then $r/q = (q/r)^{-1} = 2^{-n}$. R is transitive, since if $q/r = 2^n$ and $r/s = 2^m$, then $q/s = (q/r)(r/s) = 2^{n+m}$.

(b) $[1/2] = [2] = S$ and $[3] = \{3 \cdot 2^n : n \in \mathbb{Z}\}$.

7. 3 marks Define a relation R on $\mathbb{Z}$ by aRb if and only if $a \equiv b \bmod 2$ and $a \equiv b \bmod 3$. Prove of disprove: R is an equivalence relation on $\mathbb{Z}$.

Solution:

R is clearly reflexive and symmetric. Suppose that aRb and bRc. Then $b = 2k + a = 3\ell + a$ and $c = 2m + b = 3n + b$ for some $k, \ell, m, n \in \mathbb{Z}$. Hence $c = 2(k + m) + b = 3(\ell + n) + b$, and so aRc. Therefore R is an equivalence relation.

8. 3 marks The relation R on $\mathbb{Z}$ defined by aRb if and only if $a^3 \equiv b^3 \bmod 4$ is an equivalence relation (Verify this on your own, but do not hand it in for marks). Determine the distinct equivalence classes.

Solution:

Note that aRb if and only if $a^3 - b^3 \equiv 0 \bmod 4$.

For any $a \in \mathbb{Z}$, $a \equiv 0, 1, 2$ or $3 \bmod 4$. Therefore, since $0^3 \equiv 0$, $1^3 \equiv 1$, $2^3 \equiv 0$, $3^3 \equiv 1 \bmod 4$, there are two distinct equivalence classes: $[0] = \{4n, 4n+2 : n \in \mathbb{Z}\}$ and $[1] = \{4n+1, 4n+3 : n \in \mathbb{Z}\}$.

9. [4 marks] Let

$$A = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a + b\sqrt{2} \neq 0\}.$$

Let R be the relation on A defined by xRy if and only if $x/y \in \mathbb{Q}$. Then R is an equivalence relation (Verify this on your own, but do not hand it in for marks). Determine the distinct equivalence classes.

Solution:

Let $a, b, c, d \in \mathbb{Q}$. Note that $c + d\sqrt{2} \in [a + b\sqrt{2}]$ if for some $n, m \in \mathbb{Z} - \{0\}$ we have that $n(a + b\sqrt{2}) = m(c + d\sqrt{2})$. This implies that $na = mc$ and $nb = md$. If $a = 0$, then since $n, m \neq 0$, we have that $c = 0$. On the other hand, if $a \neq 0$, then $n = mc/a$ and so $nb = md \Rightarrow d = bc/a$.

The distinct equivalence classes are therefore $[\sqrt{2}] = \{b\sqrt{2} : b \in \mathbb{Q}\}$ and, for each $b \in \mathbb{Q}$, $[1 + b\sqrt{2}] = \{c(1 + b\sqrt{2}) : c \in \mathbb{Q}\}$.

10. [5 marks] The *factorial* of a positive integer n is defined by

$$n! = 1 \cdot 2 \cdot \cdots \cdot n$$

Prove that, for every positive integer n, $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$.

2. Prove that every positive integer m can be written in 'base factorial', i.e. in the form

$$m = a_1 \cdot 1! + a_2 \cdot 2! + \ldots + a_n \cdot n!$$

for some positive integer n and integers $a_1, \ldots, a_n$ which have the property that $0 \leq a_i \leq i - 1$ for each $i \in \{1, \ldots, n\}$.