1. **(10.3)** Let A and B be disjoint denumerable sets. Prove that $A \cup B$ is denumerable.

2. **(10.16)** Let $A_1, A_2, A_3, \ldots$ be disjoint denumerable sets. Prove that $\bigcup_{n=1}^{\infty} A_n$ is denumerable.

3. Let A be a denumerable set. Prove that the disjoint union

$$\bigcup_{n=1}^{\infty} A^{\times n} = A \cup (A \times A) \cup (A \times A \times A) \cup \cdots$$

is denumerable. (*Hint:* use the previous question.)

4. **(10.5)** Prove that $|\mathbb{Z}| = |\mathbb{Z} - \{2\}|$.

5. **(10.19)** Prove that every denumerable set can be partitioned into a denumerable number of denumerable sets.

6. (a) Prove that the set S of all numbers of the form $\sqrt[n]{a}$, where $n \in \mathbb{N}$ and $a \in \mathbb{Q}$, is countable.
   (b) Use question (3) to prove that the set of all numbers formed by finite sums of elements of S, is countable.
   (c) **(10.20)** Prove that the set of irrational numbers is uncountable. You may assume the fact that the set of real numbers is uncountable.[1]

7. **(10.25)** Consider the function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-x^2}$. Prove that f is a bijection.

8. **(10.31)** Prove that there is no set A such that $|A| = |\mathcal{P}(A)|$. (*Hint:* Suppose there is a surjective function $f : A \to \mathcal{P}(A)$. Now construct an element of $\mathcal{P}(A)$ which is not in the image of f.)

9. Prove that if there is a bijection from A to B, then there is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

10. **(10.32)** Let A, B, C be nonempty sets such that $A \subseteq B \subseteq C$ and $|A| = |C|$. Use the Schroder-Bernstein theorem to show that $|A| = |B|$ and $|B| = |C|$.

[1] This shows that most irrational numbers cannot be built from roots. In fact, the set of real numbers which appear as roots of polynomials with rational coefficients (called *algebraic numbers*) is countable. Therefore, most irrational numbers can't even be built with polynomials!

11. **(Optional, not for points)** In this multi-part question, you'll prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

(a) First, show that the function $f : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ given by

$$f(r) = \{q \in \mathbb{Q} : q < r\}$$

is injective. Use this, along with question (9), to argue that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.

(b) Next, show that the function $g : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ given by

$$g(S) = \sum_{n \in S} \frac{1}{3^n}$$

is injective. Use this to argue that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

(c) Finally, use the Schroder-Bernstein Theorem to prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.