1. **(10.3)** Let A and B be disjoint denumerable sets. Prove that $A \cup B$ is denumerable.

Proof. By assumption, there are bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} rightarrow B$. That is, one can list out the elements

$$A = \{f(1), f(2), f(3), \ldots\}$$

$$B = \{g(1), g(2), g(3), \ldots\}$$

Then one can list out the elements of $A \cup B$

$$A \cup B = \{f(1), g(1), f(2), g(2), f(3), g(3), \ldots\}$$

namely there's a bijection $h : \mathbb{N} \to A \cup B$ given by $h(2n-1) = f(n)$ and $h(2n) = g(n)$ for every $n \in \mathbb{N}$. □

2. **(10.16)** Let $A_1, A_2, A_3, \ldots$ be disjoint denumerable sets. Prove that $\bigcup_{n=1}^{\infty} A_n$ is denumerable.

Proof 1. By assumption, there are bijections $f_n : \mathbb{N} \to A_n$ for every $n \in \mathbb{N}$. Thus, one obtains a bijection

$$f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

$$(n, m) \mapsto f_n(m)$$

Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection: for example $g(i, j) = 2^{i-1}(2j-1)$. It has an inverse $g^{-1} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then consider the function

$$f \circ g^{-1} : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

This is a composition of two bijections, and therefore is a bijection. □

Proof 2. Let the elements of these sets be

$$A_1 = \{a_{1,1}, a_{1,2}, a_{1,3}, \ldots\}$$

$$A_2 = \{a_{2,1}, a_{2,2}, a_{2,3}, \ldots\}$$

$$A_3 = \{a_{3,1}, a_{3,2}, a_{3,3}, \ldots\}$$

$$\vdots$$

Then one may denumerate $\bigcup_{n=1}^{\infty} A_n$ along the diagonals

$$\bigcup_{n=1}^{\infty} A_n = \{a_{1,1}, a_{2,1}, a_{1,2}, a_{3,1}, a_{2,2}, a_{1,3}, a_{4,1}, a_{3,2}, a_{2,3}, a_{1,4}, \ldots\}$$

□

3. Let A be a denumerable set. Prove that the disjoint union

$$\bigcup_{n=1}^{\infty} A^{\times n} = A \cup (A \times A) \cup (A \times A \times A) \cup \cdots$$

is denumerable. (*Hint:* use the previous question.)

Proof. By the result of Question 2, if we can show that $A^{\times n}$ is denumerable for every n, then it will follow that $\bigcup_{n=1}^{\infty} A^{\times n}$ is denumerable. We will show by induction on n that $A^{\times n}$ is denumerable.

- Base case: $n = 1$. By assumption, A is denumerable.
- Inductive step: Assume that $A^{\times n}$ is denumerable. Then there is a bijection $f_n : \mathbb{N} \to A^{\times n}$. By assumption, A is denumerable, so there is a bijection $f : \mathbb{N} \to A$. A Cartesian product of bijections is a bijection, and so

$$f_n \times f : \mathbb{N} \times \mathbb{N} \to A^{\times n} \times A = A^{\times(n+1)}$$

is bijective. As in Q2, let $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be the bijection $g(i, j) = 2^{i-1}(2j-1)$. Then g^{-1} is the inverse bijection, and the composition of two bijections is a bijection, so

$$(f_n \times f) \circ g^{-1} : \mathbb{N} \to A^{\times(n+1)}$$

is a bijection. This completes the induction.

□

4. **(10.5)** Prove that $|\mathbb{Z}| = |\mathbb{Z} - \{2\}|$.

Proof. Consider the function $f : \mathbb{Z} \to \mathbb{Z} - \{2\}$ given by

$$f(x) = \begin{cases} x + 1 & x \geq 2 \\ x & x \leq 1 \end{cases}$$

We claim f is bijective. To show f is injective, it suffices to show that if $x < y$, then $f(x) < f(y)$. This is clear from the definition. To show this function is surjective, we must show that for every y, there exists some x such that $f(x) = y$. If $y \leq 1$, take $x = y$, and if $y \geq 3$, take $x = y - 1$. □

5. **(10.19)** Prove that every denumerable set can be partitioned into a denumerable number of denumerable sets.

Proof. Let A be any denumerable set. Then there is a bijection $f : \mathbb{N} \to A$. Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be your favorite bijection (for example, $g(i, j) = 2^{i-1}(2j - 1)$). Then $f \circ g : \mathbb{N} \times \mathbb{N} \to A$ is a bijection. For each $n \in \mathbb{N}$, let A_n be the image

$$A_n = (f \circ g)(\{n\} \times \mathbb{N})$$

Because $f \circ g$ is injective, the sets $A_1, A_2 \ldots$ are disjoint, and because $f \circ g$ is surjective, their union equals A. Therefore, $A_1, A_2, \ldots$ form a partition of A. Since $\{n\} \times \mathbb{N}$ bijects with $\mathbb{N}$, each A_n is denumerable. We have therefore partitioned A into a denumerable number of denumerable sets. □

6. (a) Prove that the set S of all numbers of the form $\sqrt[n]{a}$, where $n \in \mathbb{N}$ and $a \in \mathbb{Q}$, is countable.

Proof. By the definition of S, we have a surjection

$$f : \mathbb{N} \times \mathbb{Q} \to S$$

$$(n, a) \mapsto \sqrt[n]{a}$$

We also have a surjection $p : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $p(a, b) = \frac{a}{b}$. Thus, the composition $f \circ (\mathrm{id}_{\mathbb{N}} \times p)$ gives a surjection $\mathbb{N} \times \mathbb{Z} \times \mathbb{N} \to S$. Composing with a bijection $\mathbb{N} \to \mathbb{Z}$, we get a surjection $\mathbb{N}^{\times 3} \to S$. Then composing with a bijection $\mathbb{N} \to \mathbb{N}^{\times 3}$ (shown to exist in Question 3), we get a surjection

$$h : \mathbb{N} \to S$$

From here, we can define an injective function $i : S \to \mathbb{N}$ by letting $i(s)$ be the least element of $f^{-1}(s)$, for every $s \in S$. This puts S in bijection with a subset of $\mathbb{N}$, and therefore S is countable. (In fact, since S is not a finite set, it must be denumerable - this is because every infinite subset of $\mathbb{N}$ is in bijection with $\mathbb{N}$.) □

(b) Use question (3) to prove that the set of all numbers formed by finite sums of elements of S, is countable.

Proof. Let T be the set described, i.e.

$$T = \left\{ x \in \mathbb{R} : \exists a_1, \ldots, a_k \in \mathbb{Q}, \exists n_1, \ldots, n_k \in \mathbb{N}, x = \sum_{i=1}^{k} \sqrt[n_i]{a_i} \right\}$$

For every $k \in \mathbb{N}$, you have a function $f_k : S^{\times k} \to T$ given by

$$(\sqrt[n_1]{a_1}, \ldots, \sqrt[n_k]{a_k}) \mapsto \sqrt[n_1]{a_1} + \ldots + \sqrt[n_k]{a_k}$$

Putting these together gives a function

$$f : \bigcup_{k=1}^{\infty} S^{\times k} \to T$$

which is surjective, by the definition of T. However, since S is denumerable, $\bigcup_{k=1}^{\infty} S^{\times k}$ is denumerable (Q3). Then we have a bijection $\mathbb{N} \to \bigcup_{k=1}^{\infty} S^{\times k}$, and composing with the above function f gives a surjection $\mathbb{N} \to T$. By reasoning similar to that in (a), T is denumerable.

□

(c) **(10.20)** Prove that the set of irrational numbers is uncountable. You may assume the fact that the set of real numbers is uncountable.[1]

7. **(10.25)** Consider the function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-x^2}$. Prove that f is a bijection.

Proof. Suppose, for a contradiction, that the set $\mathbb{I}$ of irrational numbers is countable. We have already shown that $\mathbb{Q}$ is denumerable. Thus, $\mathbb{I} \cup \mathbb{Q}$ is denumerable. But $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$, and $\mathbb{R}$ is uncountable. We have therefore reached a contradiction! Thus, $\mathbb{I}$ is uncountable. □

Proof. First, we prove f is injective. Suppose that $f(x) = f(y)$ for some $x, y \in (-1, 1)$. Then

$$\begin{aligned} \frac{x}{1-x^2} = \frac{y}{1-y^2} &\implies x(1-y^2) = y(1-x^2) \\ &\implies x - y = xy^2 - x^2y = -xy(x-y) \end{aligned}$$

Therefore, either $x = y$ or $xy = 1$. But since $|x| < 1$ and $|y| < 1$, $|xy| < 1$ and so it is impossible to have $xy = 1$. Therefore, $x = y$. So we have shown f is injective.

Now we prove f is surjective. Let r be any real number. We must show there is some $x \in (-1, 1)$ such that $f(x) = r$. If $r = 0$, we can pick $x = 0$. If we can show the result for any $r > 0$, then the result will hold for any $r < 0$ because $f(-x) = -f(x)$. So assume $r > 0$.

$$\begin{aligned} \frac{x}{1-x^2} = r &\iff x = r - rx^2 \\ &\iff rx^2 + x - r = 0 \\ &\iff x = \frac{-1 \pm \sqrt{1+4r^2}}{2r} \end{aligned}$$

Thus, $x = \frac{-1+\sqrt{1+4r^2}}{2r}$ satisfies $f(x) = r$. It now suffices to show that $x \in (-1, 1)$. Indeed,

$$0 = \frac{-1+1}{2} < \frac{-1+\sqrt{1+4r^2}}{2r} < \frac{-1+\sqrt{1+4r+4r^2}}{2r} = \frac{-1+(1+2r)}{2r} = 1$$

and so $0 < x < 1$. □

8. **(10.31)** Prove that there is no set A such that $|A| = |\mathcal{P}(A)|$. (*Hint:* Suppose there is a surjective function $f : A \to \mathcal{P}(A)$. Now construct an element of $\mathcal{P}(A)$ which is not in the image of f.)

Proof. We will show that there can be no surjective function $A \to \mathcal{P}(A)$. This will necessarily imply there can be no bijective function, and so $|A| \neq |\mathcal{P}(A)|$.

[1] This shows that most irrational numbers cannot be built from roots. In fact, the set of real numbers which appear as roots of polynomials with rational coefficients (called *algebraic numbers*) is countable. Therefore, most irrational numbers can't even be built with polynomials!

Suppose, for a contradiction, that there is a surjective function $f : A \to \mathcal{P}(A)$. Consider the following subset of A.

$$S = \{a \in A : a \notin f(a)\}$$

We claim that S is not in the image of f. Suppose, for a contradiction, that there is some $x \in A$ such that $f(x) = S$. Then

- If $x \in S$, then $x \in f(x)$ and so, by the definition of S, $x \notin S$. Contradiction!
- If $x \notin S$, then $x \notin f(x)$ and so, by the definition of S, $x \in S$. Contradiction!

Therefore, there can be no x such that $f(x) = S$. Thus, S is not in the image of f.

This contradicts the assumption that f is surjective. Therefore, no surjective function $f : A \to \mathcal{P}(A)$ exists. □

Note: This proof might be easier to understand if you think of $\mathcal{P}(A)$ as the set of functions from A to $\{0, 1\}$!

9. Prove that if there is a bijection from A to B, then there is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

Proof. Let $f : A \to B$ be the bijection. We then have a function $\mathcal{P}(A) \to \mathcal{P}(B)$ which we will call $f_{\mathcal{P}}$ - it is defined by

$$f_{\mathcal{P}}(S) = f(S)$$

Let $g : B \to A$ be the inverse of f. We may similarly define $g_{\mathcal{P}} : \mathcal{P}(B) \to \mathcal{P}(A)$ by

$$g_{\mathcal{P}}(T) = g(T)$$

We claim that $f_{\mathcal{P}}$ and $g_{\mathcal{P}}$ are inverses. We must check that the two possible compositions are both identity functions.

- $g_{\mathcal{P}} \circ f_{\mathcal{P}} = \mathrm{id}_{\mathcal{P}(A)}$: We must check that $g(f(S)) = S$ for any subset $S \subseteq A$. For any $x \in S$, $g(f(x)) = x$, and so this is clear.
- $f_{\mathcal{P}} \circ g_{\mathcal{P}} = \mathrm{id}_{\mathcal{P}(B)}$: This is similar to the previous case.

□

10. **(10.32)** Let A, B, C be nonempty sets such that $A \subseteq B \subseteq C$ and $|A| = |C|$. Use the Schroder-Bernstein theorem to show that $|A| = |B|$ and $|B| = |C|$.

Proof. Because $|A| = |C|$, there is a bijective function $f : C \to A$. Let $i : A \to B$ and $j : B \to C$ be the inclusions.

- $f \circ j : B \to A$ is a composition of two injections and is therefore an injection. Since we have injections $A \to B$ and $B \to A$, the Schroder-Bernstein theorem tells us that $|A| = |B|$.

- $i \circ f : C \to B$ is a composition of of two injections and is therefore an injection. Since we have injections $B \to C$ and $C \to B$, the Schroder-Bernstein theorem tells us that $|B| = |C|$.

□

11. **(Optional, not for points)** In this multi-part question, you'll prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

(a) First, show that the function $f : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ given by

$$f(r) = \{q \in \mathbb{Q} : q < r\}$$

is injective. Use this, along with question (9), to argue that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.

Proof. Let's show that f is injective. Suppose that $r < s$ are two real numbers. Then, from a previous homework assignment, there exists a rational number q such that $r < q < s$. Then $q \in f(s)$ but $q \notin f(r)$. Thus, $f(r) \neq f(s)$. We have shown f is injective.

Since there is a bijection $\mathbb{Q} \to \mathbb{N}$, there is (by Q9) a bijection $\mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{N})$. Composing with f, we get an injection $\mathbb{R} \to \mathcal{P}(\mathbb{N})$. Thus, $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$. □

(b) Next, show that the function $g : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ given by

$$g(S) = \sum_{n \in S} \frac{1}{3^n}$$

is injective. Use this to argue that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Proof. Let S and T be two distinct subsets of $\mathbb{N}$. Then there exists some element of $\mathbb{N}$ which is in one of the sets and not in the other. Let n be the smallest such number, and WLOG, $n \in S$, $n \notin T$.

We will show that $g(S) > g(T)$. Let U be the set of natural numbers less than n which lie in S. Because n was the smallest number which lies in one set and not the other, U is also equal to the set of natural numbers less than n which lie in T. We have

$$g(S) \geq g(U) + \frac{1}{3^n}$$

and

$$\begin{aligned} g(T) &\leq g(U) + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \frac{1}{3^{n+3}} + \dots \\ &= g(U) + \frac{1}{2 \cdot 3^n} \end{aligned}$$

Thus, $g(S) \geq g(U) + \frac{1}{\cdot 3^n} > g(U) + \frac{1}{2 \cdot 3^n} \leq g(T)$, and so $g(S) > g(T)$.

So we have shown that if S, T are distinct subsets of $\mathbb{N}$, then $g(S) \neq g(T)$. Hence, $g : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is injective, so $|\mathcal{P}(\mathbb{N})| \leq \mathbb{R}$. □

(c) Finally, use the Schroder-Bernstein Theorem to prove that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Proof. We've constructed an injective function $\mathbb{R} \to \mathcal{P}(\mathbb{N})$ and an injective function $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$. Therefore, by the Schroder-Bernstein theorem, $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$. (Note that the two functions we constructed have nothing to do with each other!) $\square$